ISRAEL JOURNAL OF MATHEMATICS, Vol. 33, No. 1, 1979

SOME RELATIONS IN THE MOD 3 COHOMOLOGY OF H-SPACES

BY

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ABSTRACT

A third order non stable operation is constructed. Its domain and range are a linear subspace and a quotient space of the mod 3 cohomology but it represents a non linear transformation. Using this operation some relations over the Steenrod algebra in the module of primitives in the cohomology of an H-space are derived even in the 3 torsion free case.

The study of non stable mod p operations proved to be useful in the investigation of the cohomology of H-spaces (see e.g. Thomas [11], Browder [1], [2], Hubbuck [4], Zabrodsky [12], Lin [6]). The operations studied were either high order Bockstein operations or secondary operations induced by relations in the (non stable) Steenrod algebra following from

$$\text{excess } \mathcal{P}^n > 2n - 1, \tag{R$_1$}$$

$$\text{excess } \beta\mathcal{P}^n > 2n. \tag{R$_2$}$$

For $p > 2$ $(\mathrm{R})_1$ was useful only when some associativity properties were available (e.g. for $p = 3$ homotopy associativity was essential). The higher Bockstein operations and those related to $(\mathrm{R})_2$ provided information involving even dimensional generators and torsion. It seemed that very little could be said about a non associative mod odd H-space with exterior cohomology algebra. The fact that products of odd spheres are mod odd H-spaces supported this view: At least no limitations on the type of a mod odd H-space exist.

The first evidence that some relations in the cohomology of a mod odd H-space must be present was discovered by Mimura and Toda (see Mimura [8]).

Received May 4, 1978

They proved that $B_9(3)$ and $B_{15}(3)$ (see Mimura [7]) are not mod 3 H-spaces although $B_n(p)$'s, $p>2$, are all π-power spaces (in the sense of Zabrodsky [14]).

The purpose of this paper is to give a cohomology operations interpretation to these results of Mimura and Toda whose paper, needless to say, strongly motivated this study. I would like to thank John Harper for suggesting the possible existence of a general pattern related to the Mimura–Toda results.

In some sense this note complements the results in Harper–Zabrodsky [3] showing them to be the best possible for $p=3$. We suspect similar results hold for $p>3$ but involve p-th order operations.

The main object of study in this paper is a third order operation Φ with a universal example $(\hat{x}, E, y)$ where E is an H-space (with three non vanishing homotopy groups), $\hat{x}$-primitive while y is a non primitive generator. Hence for an arbitrary space domain $\Phi \subset H^{|x|}(X, Z_3)$ and target Φ are linear spaces while Φ is not a linear transformation. More precisely:

THEOREM 1. *Let a, n be positive integers, $1 \leqq a \leqq 2$, $\binom{2n-1}{a} \not\equiv 0 \pmod 3$. Consider the relations*

$$\text{(r)} \qquad \text{excess}\, \mathcal{P}^a\mathcal{P}^n > 2(n+a)-1, \qquad \mathcal{P}^{3-a}\mathcal{P}^a = 0.$$

Then

(i) $0 = \langle \mathcal{P}^{3-a}, \mathcal{P}^a, \mathcal{P}^n \rangle$ *as an operation on* $\leqq 2(n+a)-1$ *dimensional classes.*

(ii) (i) *induces a third order operation ϕ_a with a universal example*: $(\hat{x}, E, y)$, E — *an H-space $\hat{x} \in PH^{2n+2a-1}(E, Z_3)$ and y is a non primitive generator with reduced coproduct*

$$(a=2) \qquad \bar{\mu}^* y = \mathcal{P}^1\hat{x}\cdot\hat{x} \otimes \hat{x} + \hat{x} \otimes \mathcal{P}^1\hat{x}\cdot\hat{x},$$

$$(a=1) \quad \bar{\mu}^* y = \hat{x}\cdot\mathcal{P}^1\hat{x} \otimes \mathcal{P}^1\hat{x} + \mathcal{P}^1\hat{x} \otimes \hat{x}\cdot\mathcal{P}^1\hat{x} + \mathcal{P}^2\hat{x}\cdot\hat{x} \otimes \hat{x} + \hat{x} \otimes \mathcal{P}^2\hat{x}\cdot\hat{x}.$$

Hence if $z_1, z_2 \in H^{2n+2a-1}(X, Z_3)$ are in domain Φ_a *so is $z_1 + z_2$ and*

$$(a=2) \qquad \phi_a(z_1+z_2) = \phi_a(z_1) + \phi_a(z_2) + \mathcal{P}^1 z_1\cdot z_1\cdot z_2 + z_1\cdot\mathcal{P}^1 z_2\cdot z_2,$$

$$(a=1) \qquad \begin{aligned} \phi_a(z_1+z_2) = \phi_a(z_1) + \phi_a(z_2) + z_1\cdot\mathcal{P}^1 z_1\cdot\mathcal{P}^1 z_2 + \mathcal{P}^1 z_1\cdot z_2\cdot\mathcal{P}^1 z_2 \\ + \mathcal{P}^2 z_1\cdot z_1\cdot z_2 + z_1\cdot\mathcal{P}^2 z_2\cdot z_2. \end{aligned}$$

The applications to mod 3 H-spaces are illustrated by:

THEOREM 2. *Let X be a* mod 3 *H-space with primitively generated rational cohomology and $H^*(X, Z_3)$ an exterior algebra on odd dimensional generators. If $x \in PH^{6n+m}(X, Z_3)$, $m = 1$ or 3 and $\mathcal{P}^{3n}x = 0$ then $\mathcal{P}^1 x \in \operatorname{im}\mathcal{P}^2 x$.*

COROLLARY. *If* $n+1 \not\equiv 0$ (mod 3) *then* $B_n(3)$ *are not* mod 3 H*-spaces* (*e.g.* $O(6n+5)/O(6n+1)$ *is not a* mod 3 H*-space*).

PROOF OF COROLLARY. $H^*(B_n(3), Z_3) = \Lambda(x_{2n+1}, \mathscr{P}^1 x_{2n+1})$. If $n+1 \not\equiv 0$ (mod 3), $2n+1 = 6m+2a-1$, $a = 1, 2$ and by Theorem 2 this cannot be a cohomology of a mod 3 H-space

The paper is organized as follows: In Section 1 we compile some known facts and their immediate consequences regarding H-structures and homotopy associative and homotopy commutative invariants. We use these in the proof of Theorem 1 in Section 2. Theorem 2 is proved in Section 3.

1. H structures, homotopy associativity and homotopy commutativity invariants

We use notations and follow the proofs of Zabrodsky [13], though the notions discussed could be traced back to Kudo–Araki [5], Browder [2] and Stasheff [9], [10].

1.1. (See Zabrodsky [13] (1.18), (2.1), (2.2), (3.2).) (a) Let $f, F : X, \mu \to X', \mu'$ be an H map, $F : f \circ \mu \sim \mu' \circ (f \times f)$. Then f, F, μ, μ' induce a multiplication μ_E on $E = \text{fiber } f$. If $* \sim f$ with a homotopy $l : X \to \mathscr{L}X'$ then there exists $h : X \times \Omega X' \approx E$ and the multiplication in $X \times \Omega X'$ induced by h and μ_E is given by

$$(x, \lambda) \cdot (x', \lambda') = w(x, x') + \lambda + \lambda',$$

where $w = w(l, F) : X \wedge X \to \Omega X'$ is given by

$$w \circ \Lambda = l \circ \mu + F - \mathscr{L}\mu' \circ (l \times l) : X \times X \to \Omega X'.$$

(b) Let $\alpha : X \to \Omega X'$. Then:

$$w(\alpha + l, F) - w(l, F) = \bar{\mu}^* \alpha : X \wedge X \to \Omega X'$$

where $\bar{\mu}^* : [X, \Omega X'] \to [X \wedge X, \Omega X']$ is given by

$$\Lambda^* \circ \bar{\mu}^* = \mu^* - p_1^* - p_2^* : [X, \Omega X'] \to [X \times X, \Omega X'],$$

$$\Lambda^* : [X \wedge X, \Omega X'] \to [X \times X, \Omega X'].$$

1.2. (See Zabrodsky [13] 2.5.) Let X, μ, A and X', μ', A' be homotopy associative H-spaces (abr: HA spaces). $A:\mu\circ(\mu\times 1)\sim\mu\circ(1\times\mu)$. Let $f, F: X,\mu\to X',\mu'$ be an H-map. f, F, A, A' define an HA invariant

$$\theta(f, F, A, A')\in[X\wedge X\wedge X,\Omega X']$$

with the following porperties:

(a) Composition: Given HA spaces (X,μ,A), (X',μ',A'), (X'',μ'',A'') and H-maps

$$f, F: X,\mu\to X',\mu',$$

$$f', F', X',\mu'\to X'',\mu'',$$

then:

$$\theta((f',F')\circ(f,F),A,A'')=\Omega f'\circ\theta(f,F,A,A')+\theta(f',F',A',A'')\circ(f\wedge f\wedge f).$$

(b) Dependence on H-structure: If $v: X\wedge X\to\Omega X'$ then

$$\theta(f, v+F, A, A'')-\theta(f,F,A,A')=d_1 v: X\wedge X\wedge X\to\Omega X',$$

d_1 given by

$$\begin{aligned}\Lambda_2^*\circ d_1=\{&[(\mu\times 1)^*-(p_1\times 1)^*-(p_2\times 1)^*]\\&-[(1\times\mu)^*-(1\times p_1)^*-(1\times p_2)^*]\}\circ\Lambda^*\end{aligned}$$

$$\begin{array}{ccc}[X\wedge X,\Omega X'] & \xrightarrow{d_1} & [X\wedge X\wedge X,\Omega X']\\ \downarrow\Lambda^* & & \downarrow\Lambda_2^*\\ [X\times X,\Omega X'] & \longrightarrow & [X\times X\times X,\Omega X']\end{array}$$

(c) If $(f,F): X,\mu\to X',\mu'$ is an H-map of HA spaces X,μ,A; X',μ',A' and $\alpha: f\sim\bar f$ then $\hat f,\hat F=\bar f,\ -\alpha\circ\mu+F+PP\mu'\circ(\alpha\times\alpha)$ is an H-map and

$$\theta(f,F,A,A')=\theta(\hat f,\hat F,A,A'').$$

(d) If $f, F: X,\mu\to X',\mu'$ is actually multiplicative (i.e. $f\circ\mu=\mu'\circ(f\times f)$ and F is the constant homotopy), X,μ,A; X',μ',A' are HA and $f\sim *$ then

$$\theta(f,F,A,A')=0.$$

(e) Let $f: Y\to Y'$ be a map. Then ΩY, $\Omega Y'$ have obvious HA structures A, A' respectively and Ωf is multiplicative with a constant homotopy F. In this case

$$\theta(f,F,A,A')=0.$$

1.2.1. Lemma. *Let $f, F : X, \mu \to X', \mu'$ be an H-map of HA spaces X, μ, A and X', μ', A'. If $* \sim f$ then for any choice of homotopy $l : * \sim f$*

$$\theta(f, F, A, A') = d_1 w(l, F).$$

Proof. As in 1.2(c) l induces an H-structure $\hat{F}$ for $\hat{f} = *$ and one can easily see that $\hat{F} = w(l, F)$. By 1.2(b) and (d)

$$\theta(f, F, A, A') = \theta(*, w(l, F), A, A') = \theta(*, *, A, A') + d_1 w(l, F)$$
$$= d_1 w(l, F).$$

(The independence on choice of l follows from $d_1 w(\alpha + l, F) = d_1 \bar{\mu}^* \alpha + d_1 w(l, F)$ and $d_1 \bar{\mu}^* = 0$.)

1.3. Lemma. *Let $f : K(Z_p, 2n) \to K(Z_p, 6n)$ be given by $f^* \iota_{6n} = \iota^3_{\sigma n}$. Put $E =$* fiber f. *Then one has an H-map*

$$f', F' : K(Z_p, 2n-1) \to \Omega E \approx K(Z_p, 2n-1) \times K(Z_p, 6n-2)$$

and if A, A' are the loop A-structures of $K(Z_p, 2n-1)$ and ΩE respectively then $\theta(f', F', A, A') : K(Z_p, 2n-1) \wedge K(Z_p, 2n-1) \wedge K(Z_p, 2n-1) \to \Omega^2 E$ factors as

$$K(Z_p, 2n-1) \wedge K(Z_p, 2n-1) \wedge K(Z_p, 2n-1) \xrightarrow{\iota_{2n-1} \otimes \iota_{2n-1} \otimes \iota_{2n-1}} K(Z_p, 6n-3) \to \Omega^2 E.$$

Proof. In view of 1.1(a) the existence of the H-map f', F' will follow from the existence of $l : * \sim \Omega f$ so that if $F = k$ is the constant H-structure of Ωf then $w(l, k) = 0$. By Zabrodsky [13] 3.2.2 one has a natural homotopy $\hat{l} : * \sim \Omega \bar{\Delta}$ so that if k is the constant H-structure of the multiplicative map

$$\Omega \bar{\Delta} : \Omega K(Z_p, 2n) \to \Omega(K(Z_p, 2n) \wedge K(Z_p, 2n))$$

then the adjoint of

$$w(\hat{l}, k) : \Omega K(Z_p, 2n) \wedge \Omega K(Z_p, 2n) \to \Omega^2(K(Z_p, 2n) \wedge K(Z_p, 2n))$$

is given by

$$\Sigma^2(\Omega K(Z_p, 2n) \wedge \Omega K(Z_p, 2n)) = \Sigma\Omega K(Z_p, 2n) \wedge \Sigma\Omega K(Z_p, 2n) \xrightarrow{(\in \wedge \in) \circ T}$$

$$\longrightarrow K(Z_p, 2n) \wedge K(Z_p, 2n)$$

$(\in : \Sigma\Omega K(Z_p, 2n) \to K(Z_p, 2n)$ — the evaluation).

Now $f : K(Z_p, 2n) \to K(Z_p, 6n)$ factors as

$$K(Z_p, 2n) \xrightarrow{\bar{\Delta}} K(Z_p, 2n) \wedge K(Z_p, 2n) \xrightarrow{f_1 \wedge 1}$$

$$\longrightarrow K(Z_p, 4n) \wedge K(Z_p, 2n) \xrightarrow{u} K(Z_p, 6n),$$

$f_1^* \iota_{4n} = \iota_{2n}^2$, $u^* \iota_{6n} = \iota_{4n} \otimes \iota_{2n}$. Choose $l : * \sim \Omega f$ by

$$l = (\mathscr{L}\Omega u) \circ (\mathscr{L}\Omega(f_1 \wedge 1)) \circ \hat{l}.$$

Then one can easily see that

$$w(l, k) = \Omega^2 u \circ \Omega^2(f_1 \wedge 1) \circ w(\hat{l}, k),$$

$$[w(l, k)]_{\#} = u \circ (f_1 \wedge 1) \circ \in \wedge \in \circ T = u \circ (f_1 \circ \in \wedge \in) \circ T.$$

But as $(f_1 \circ \in)_{\#} = \Omega f_1 \sim *$, $w(l, k)_{\#} \sim * \sim w(l, k)$. Now, if f', F': $K(Z_p, 2n-1) \to \Omega E$ satisfies $\theta(f, F, A, A') = 0$ then one has a map

$$f_3' : B_3(K(Z_p, 2n-1)) \to E$$

(B_3—the third stage of the classifying space of $K(Z_p, 2n-1)$, see Stasheff [10]), $\pi_*(f_3')$-injective. Now $(K(Z_p, 2n), B_3(K(Z_p, 2n-1)))$ is $8n-1$ connected, $\pi_k(E) = 0$ for $k \geqq 6n$ and there are no obstructions to extend f_3' to $f_\infty' : k(Z_p, 2n) \to E$, $\pi_*(f_\infty')$-injective which will yield the contradiction:

$$E \approx K(Z_p, 2n) \times K(Z_p, 6n-1) \quad \text{or} \quad \iota_{2n}^3 = 0.$$

Hence, $0 \neq \theta(f, F, A, A') : K(Z_p, 2n-1) \wedge K(Z_p, 2n-1) \wedge K(Z_p, 2n-1) \to \Omega^2 E$. This map lifts (in a unique way) to

$$0 \neq \tilde{\theta} : K(Z_p, 2n-1) \wedge K(Z_p, 2n-1) \wedge K(Z_p, 2n-1) \longrightarrow$$

$$\to K(Z_p, 6n-3) = \Omega^2 K(Z_p, 2n-1).$$

Hence,

$$\tilde{\theta}^* \iota_{6n-3} = \lambda(\iota_{2n-1} \otimes \iota_{2n-1} \otimes \iota_{2n-1}), \qquad 0 \neq \lambda \in Z_p.$$

One can change the representation of $E \approx K(Z_p, 2n-1) \times K(Z_p, 6n-2)$ by self equivalence so that $\lambda = 1$.

1.4. Let x, μ, C and X', μ', C' be homotopy commutative H-spaces. (abr. HC spaces. $C : \mu \sim \mu \circ T$). If $f, F : X, \mu \to X', \mu'$ is an H-map one has an invariant

$$\gamma(f, F, C, C'): X \wedge X \to \Omega X'$$

so that:

(a) Composition: If X'', μ'', C'' is a third HC space, $f', F': X', \mu' \to X'', \mu''$ an H-map then:

$$\gamma((f', F') \circ (f, F), C, C'') = \Omega\varphi' \circ \gamma(f, F, C, C') + \gamma(f', F', C', C'') \circ (f \wedge f).$$

(b) Dependence on H-structure: If $\alpha : X \wedge X \to \Omega X'$ then

$$\gamma(f, \alpha + F, C, C') - \gamma(f, F, C, C'') = (1 - T^*)\alpha,$$

$$T : X \wedge X \to X \wedge X \text{—the twisting,} \qquad T(x, y) = y, x.$$

(c) If $f, F : X, \mu \to X', \mu'$ satisfies $f \sim *$ then

$$\gamma(f, F, C, C') = (1 - T^*)w(l, F) \quad \text{for any choice of } l : * \sim f.$$

1.5. LEMMA. *Let X, μ, C; X', μ', C'; X'', μ'', C'' be HC spaces, $f, F : X, \mu \to X', \mu'$, $f', F' : X', \mu' \to X'', \mu''$ H-maps and $\gamma(f', F', C', C'') = 0$.*

Suppose the elements of $[X \wedge X, \Omega X']$ can be halved and multiplication by 2 is injective in $[X \wedge X, \Omega X'']$. If $f' \circ f \sim *$ then F can be chosen so that

$$\gamma((f', F') \circ (f, F), C, C'') = 0.$$

PROOF. Start off with an arbitrary H-structure $\hat{F}$ for f. Now choose

$$F = -\tfrac{1}{2}\gamma(f, \hat{F}, C, C') + \hat{F}.$$

By 1.4(b)

$$\begin{aligned} \gamma(f, F, C, C') &= \gamma(f, \hat{F}, C, C') - (1 - T^*)\tfrac{1}{2}\gamma(f, \hat{F}, C, C') \\ &= (1 + T^*)\tfrac{1}{2}\gamma(f, \hat{F}, C, C'). \end{aligned}$$

By 1.4(c), if $f' \circ f, \tilde{F} = (f', F') \circ (f, F)$ then

$$\gamma((f', F') \circ (f, F), C, C'') = (1 - T^*)w(l, \tilde{F}).$$

Finally by 1.4(a) and as $\gamma(f', F', C', C'') = 0$

$$\begin{aligned} \gamma[(f', F') \circ (f, F), C, C''] &= \Omega f' \gamma(f, F, C, C') \\ &= (1 + T^*)[\tfrac{1}{2}\Omega f' \gamma(f, \hat{F}, C, C')]. \end{aligned}$$

As multiplication by 2 is injective in $[X \wedge X, \Omega X'']$ one can easily see that $\operatorname{im}(1 + T^*) \cap \operatorname{im}(1 - T^*) = 0$ and $\gamma((f', F') \circ (f, F), C, C'') = 0$.

2. Proof of Theorem 1

Consider the following commutative diagram of ∞-loop spaces and maps:

$$(\mathrm{A})_N \qquad \begin{tikzcd}
K(Z_3, N+4n-1) \arrow[r, "\mathscr{P}^a"] \arrow[d] & K(Z_3, N+4n+4a-1) \arrow[d] \arrow[dr] & \\
E_1 = \mathrm{fiber}(\mathscr{P}^n) \arrow[r, "h"] \arrow[d, "r_1"] & \hat{E}_1 = \mathrm{fiber}\, \mathscr{P}^{n+a} \arrow[r, "\hat{v}"] \arrow[d] & K(Z_3, N+4n+11) \arrow[d, "\hat{r}_1"] \\
K(Z_3, N) \arrow[r, "1"] \arrow[d, "(\pm \mathscr{P}^n)"] & K(Z_3, N) \arrow[r, hook, "v"] \arrow[d] & E_0 = \mathrm{fiber}(\mathscr{P}^{3-a}) \\
K(Z_3, N+4n) \arrow[r, "\mathscr{P}^a"] & K(Z_3, N+4n+4a) \arrow[r, "\mathscr{P}^{3-a}"] \arrow[ur] & K(Z_3, N+4n+12)
\end{tikzcd}$$

As $\mathscr{P}^{3-a}\mathscr{P}^a = 0$, $\hat{v} \circ h$ factors as $w \circ r_1$ for some

$$w : K(Z_3, N) \to K(Z_3, N+4n+11).$$

For $N = 2n + 2a - 1$, $\hat{E}_1 \approx K(Z_3, 2n+2a-1) \times K(Z_3, 6n+6a-2)$ and by 1.3 this is an H-equivalence with respect to the ∞-loop multiplication of $\hat{E}_1$ and the product multiplication on $K(Z_3, 2n+2a-1) \times K(Z_3, 6n+2a-2)$.

If $i, F : K(Z_3, 2n+2a-1) \to K(Z_3, 2n+2a-1) \times K(Z_3, 6n+6a-2) \approx \hat{E}_1$ is an H-map A, A' the ∞-Ω HA structures of $K(Z_3, 2n+2a-1)$ and $\hat{E}_1$ respectively, then by 1.3 we know $\theta(i, F, A, A')$: If $K_0 = K(Z_3, 2n+2a-1)$ then

$$(\mathrm{B}) \qquad \begin{tikzcd}
K_0 \wedge K_0 \wedge K_0 \arrow[r, "\iota \otimes \iota \otimes \iota"] \arrow[d, "\theta(i) = \theta(i, F, A, A')"] & K(Z_3, 6n+6a-3) \arrow[d, "\Omega \hat{j}_1"] \\
\Omega \hat{E}_1 \approx \Omega K_0 \times & K(Z_3, 6n+6a-3).
\end{tikzcd}$$

Consider the following part of diagram $(\mathrm{A})_N$ for $N = 2n + 2a - 1$:

$$\begin{tikzcd}
 & K(Z_3, 6n+6a-2) \arrow[d, "\hat{j}_1"] \arrow[dr, phantom, "\mathscr{P}^{3-a}"] & \\
E_1 \arrow[r, "h"] & \hat{E}_1 \approx K(Z_3, 2n+2a-1) \times K(Z_3, 6n+6a-2) \arrow[r, "\hat{v}"] & K(Z_3, 6n+2a+10)
\end{tikzcd}$$

Then:

$$\hat{v}^* \iota_{6n+2a+10} = z \otimes 1 + 1 \otimes \mathscr{P}^{3-a} \iota_{6n+6a-2},$$

$z \in PH^{6n+2a+10}(K(Z_3, 2n+2a-1), Z_3)$. Let z denote its stable class as well:

$$z \in PH^{N+4n+11}(K(Z_3, N), Z_3).$$

Replacing v by $v - \hat{r}_1 z$ if necessary (and altering $\hat{v}$ and w accordingly) one may actually assume that $z = 0$ and for $N = 2n + 2a - 1$

$$\hat{v}^* \iota = 1 \otimes \mathcal{P}^{3-a} \iota.$$

If $N = 2n - 1$ one has $E_1 \approx K(Z_3, 2n-1) \times K(Z_3, 6n-2)$, $\hat{E}_1 \approx K(Z_3, 2n-1) \times K(Z_3, 6n+4a-2)$, $h^*(1 \times \iota_{6n+4a-2}) = y \otimes 1 + 1 \otimes \mathcal{P}^a \iota_{6n-2}$, $y \in PH^{6n+4a-2}(K(Z_3, 2n-1), Z_3)$ and $h^* \circ \hat{v}^* \iota = h^*(1 \otimes \mathcal{P}^{3-a} \iota_{6n+4a-2}) = \mathcal{P}^{3-a} y \otimes 1$.

Obviously $\sigma^{N-2n+1} w = \mathcal{P}^{3-a} y$. Let $\hat{y} \in H^{N+4n+4a-1}(K(Z_3, N), Z_3)$ denote the stable class of y: $\sigma^{N-2n+1} \hat{y} = y$. Then $w - \mathcal{P}^{3-a} \hat{y} \in \ker \sigma^{N-2n+1}$. But σ^{N-2n+1}: $H^{N+4n+11}(K(Z_3, N), Z_3) \to H^{6n+10}(K(Z_3, 2n-1), Z_3)$ is injective, hence, $w = \mathcal{P}^{3-a} \hat{y}$. Now alter h in $(\mathrm{A})_N$ by $-\hat{j}_1 \circ \hat{y} \circ r_1$ if necessary and as

$$\hat{v}(h - \hat{j}_1 \circ \hat{y} \circ r_1) = \hat{v} \circ h - \hat{v} \circ \hat{j}_1 \circ \hat{y} \circ \hat{r}_1 = w \circ r_1 - \mathcal{P}^{3-a} \hat{y} \circ r_1 = 0$$

one could have assumed $\hat{v} \circ h \sim *$. Put $x_1 = r_1^* \iota_{2n+2a-1} \in H^{2n+2a-1}(E_1, Z_3)$, $y_1 = h^*(1 \otimes \iota_{6n+6a-2}) \in H^{6n+6a-2}(E_1, Z_3)$ then $\hat{\phi}_1 = (x_1, E_1, y_1)$ defines an operation with domain $\ker \mathcal{P}^n \mid H^{2n+2a-1}(\ , Z_3)$.

$\hat{v}^* \iota_{6n+2a+10} = 1 \otimes \mathcal{P}^{3-a} \iota_{6n+6a-2}$ and $\hat{v} \circ h \sim *$ imply $\mathcal{P}^{3-a} y_1 = 0$, $0 = \langle \mathcal{P}^{3-a}, \mathcal{P}^a, \mathcal{P}^n \rangle$ and (i) follows.

$\hat{v} \circ h \sim *$ implies that h factors through fiber $\hat{v}$ which for $N = 2n + 2a - 1$ is equivalent to

$$K(Z_3, 2n + 2a - 1) \times \tilde{E},$$

$$\tilde{E} = \text{fiber}\,[(\mathcal{P}^{3-a}) : K(Z_3, 6n + 6a - 2) \to K(Z_3, 6n + 2a + 10)].$$

Now the operation $\phi = \phi_a$ of Theorem 1 (ii) is defined in the space $E_2 = \text{fiber}\,[(y_1) : E_1 \to K(Z_3, 6n + 6a - 2)]$ as its universal example. An alternative and equivalent definition of E_2 can be given in terms of the following pull back diagram:

$$\begin{tikzcd}
K(Z_3, 6n+6a-3) \arrow[r, "\mathcal{P}^{3-a}"] \arrow[d, "j_2"] & K(Z_3, 6n+2a+9) \arrow[d] & \\
E_2 \arrow[r, "h_2"] \arrow[d, "r_2"] \arrow[dr, phantom, "\text{pull back}"] & K(Z_3, 2n+2a-1) \times K(Z_3, 6n+2a+9) \arrow[r, "p_1"] \arrow[d, "\times \tilde{j}"] \arrow[dr, phantom, "\text{pull back}"] & K(Z_3, 2n+2a-1) \arrow[d, "i"] \\
E_1 \arrow[r, "\hat{h}"] \arrow[rr, bend right] & K(Z_3, 2n+2a-1) \times \tilde{E} \arrow[r, "\hat{h}'"] & K(Z_3, 2n+2a-1) \times K(Z_3, 6n+6a-2)
\end{tikzcd}$$

ϕ_a then has the universal example $(\hat{x}, E_2, y)$, $\hat{x} = r_2^* r_1^* \iota_{2n+2a-1}$, $y = (p_1 \circ h_2)^*_{\iota_{2n+2a-1}}$. The bottom row consists of ∞-Ω spaces and maps and are given the corresponding H, HA and HC structures. i is an H-map and any choice of H-structure for i will yield H-structures for the spaces in the second row. Thus, the H-structure of $K(Z_3, 2n+2a-1) \times K(Z_3, 6n+2a+9)$ could be considered as induced either as a pull back or equivalently as the fiber of $\hat{v} \circ i$ whose H-structure is induced by the ∞-Ω structure of $\hat{v}$ and a choice of an H-structure for i. By 1.5 such an H-structure can be chosen so that $\gamma(\hat{v} \circ i) = 0$ and the H-structure induced on $K(Z_3, 2n+2a-1) \times K(Z_3, 6n+2a+9)$ is HC: If this H-structure is given by a twisting

$$\tilde{w} : K(Z_3, 2n+2a-1) \wedge K(Z_3, 2n+2a-1) \to K(Z_3, 6n+2a+9)$$

then $\tilde{w} \circ T \sim \tilde{w}$. By 1.2, and diagram (B), if μ is the multiplication in $K(Z_3, 2n + 2a - 1)$ then (omitting the H-structures of maps and HA structures of spaces) recalling that $\theta(\hat{v}) = 0$:

$$
\begin{aligned}
(\bar{\mu}^* \otimes 1 - 1 \otimes \bar{\mu}^*)\tilde{w} &= \theta(\hat{v} \circ i) = \theta(\hat{v})(i \wedge i \wedge i) + \Omega v \circ \theta(i) \\
&= \Omega \hat{v} \circ \theta(i) = (\Omega(\hat{v} \circ \hat{j}_1))_*(\iota \otimes \iota \otimes \iota) = \mathscr{P}^{3-a}(\iota \otimes \iota \otimes \iota) \\
&= \begin{cases} \mathscr{P}^1\iota \otimes \iota \otimes \iota + \iota \otimes \mathscr{P}^1\iota \otimes \iota + \iota \otimes \iota \otimes \mathscr{P}^1\iota & \text{for } a = 2, \\[2ex] \mathscr{P}^1\iota \otimes \mathscr{P}^1\iota \otimes \iota + \mathscr{P}^1\iota \otimes \iota \otimes \mathscr{P}^1\iota + \iota \otimes \mathscr{P}^1\iota \otimes \mathscr{P}^1\iota & \\ \quad + \mathscr{P}^2\iota \otimes \iota \otimes \iota + \iota \otimes \mathscr{P}^2\iota \otimes \iota + \iota \otimes \iota \otimes \mathscr{P}^2\iota & \text{for } a = 1. \end{cases}
\end{aligned}
$$

If $w_0 : K(Z_3, 2n+2a-1) \wedge K(Z_3, 2n+2a-1) \to K(Z_3, 6n+2a+9)$ is given by

$$
w_0 = \begin{cases} \mathscr{P}^1\iota \cdot \iota \otimes \iota + \iota \otimes \mathscr{P}^1\iota \cdot \iota & \text{for } a = 2 \\[2ex] \iota \cdot \mathscr{P}^1\iota \otimes \mathscr{P}^1\iota + \mathscr{P}^1 \otimes \iota \cdot \mathscr{P}^1\iota + \mathscr{P}^2\iota \cdot \iota \otimes \iota + \iota \otimes \mathscr{P}^2\iota \cdot \iota & \text{for } a = 1 \end{cases}
$$

then $\tilde{w} - w_0 \in \ker(\bar{\mu}^* \otimes 1 - 1 \otimes \bar{\mu}^*) \cap \ker(1 - T^*)$ which implies that $\tilde{w} - w_0 \in \operatorname{im} \bar{\mu}^*$. Hence, altering $K(Z_3, 2n+2a-1) \times K(Z_3, 6n+2a+9)$ by a self equivalence if necessary one may assume $\tilde{w} = w_0 \cdot \langle \hat{x}, E, y \rangle$ with $E = E_2$, $\hat{x} = (r_1 \circ r_2)^* \iota_{2n+2a-1}$, $y = h_2^*(1 \otimes \iota_{6n+2a+9})$ is the desired operation.

3. Proof of Theorem 2

3.1. Lemma. *Let X, μ be a* mod p *H-space with $H^*(X, Q; \mu)$-primitively generated. Suppose either X is finite dimensional* (mod p) *or $\pi_n(X) = 0$ for $n > N(X)$.*

Let λ be any integer and ϕ_λ the λ-th power map induced by μ. (For $\lambda = -1$, Φ_{-1} is the multiplicative inverse; for $\lambda > 0$,

$$\phi_\lambda = \mu \circ (\mu \times 1) \circ \cdots \circ (\mu \times 1 \times \cdots \times 1) \circ (\Delta \times 1 \times \cdots \times 1) \circ \cdots \circ (\Delta \times 1) \circ \Delta;$$

for $\lambda < 0$ take $\phi_\lambda = \phi_{-1} \circ \Phi_{-\lambda}$.)

Then there exist an H-space $\hat{X}, \hat{\mu}$ mod p *equivalent to X, μ, and a map $\hat{\phi}_\lambda : \hat{X} \to \hat{X}$ so that*

$$\begin{tikzcd} X \arrow[r, "\phi_\lambda"] \arrow[d, "\approx p"'] & X \arrow[d, "\approx p"] \\ \hat{X} \arrow[r, "\hat{\phi}_\lambda"] & \hat{X} \end{tikzcd}$$

is (homotopy) commutative and so that for some r, $(\hat{\phi}_\lambda)^{p^r}$ is an H-map. Moreover, $H^*(\hat{X}, Q; \hat{\mu}) \approx H^*(X, Q; \mu)$ as Hopf algebras.

PROOF. This is basically a special case of an H-space with identity (Zabrodsky [15] 4.5). One starts by approximating X by an H-map

$$\psi_0 : X \xrightarrow{\approx 0} \prod_j K(Z, n_j) = K_0$$

K_0 having the product multiplication μ_0. Decompose ψ_0 as

$$X \xrightarrow{\approx p} X(p) = \hat{X} \xrightarrow[\psi'']{\approx \mathbf{P} - p} K_0$$

(Zabrodsky [13] 4.3) and $\hat{X}$ then admits an H-structure $\mu(p)$ and a self map $\hat{\phi}_\lambda$ induced by μ and ϕ_λ respectively. One has a commutative diagram

$$\begin{tikzcd} \hat{X} \vee \hat{X} \arrow[r] \arrow[rrr, bend left, "\mathscr{F}"] \arrow[d, "\hat{\phi}_\lambda \vee \hat{\phi}_\lambda"] & \hat{X} \times \hat{X} \arrow[r, "\mu_0 \circ (\psi'' \times \psi'')"] \arrow[d, "\hat{\phi}_\lambda \times \hat{\phi}_\lambda"] & K_0 \arrow[d, "\lambda . 1"] & \hat{X} \arrow[l, "\psi''"'] \arrow[d, "\hat{\phi}_\lambda"] \\ \hat{X} \vee \hat{X} \arrow[r] & \hat{X} \times \hat{X} \arrow[r, "\mu_0 \circ (\psi'' \times \psi'')"] & K_0 & \hat{X} \arrow[l, "\psi''"'] \end{tikzcd}$$

As $\mu(p) : \hat{X} \times \hat{X} \to \hat{X}$ is a lifting of $\mu_0(\psi'' \times \psi'')$ one can apply Zabrodsky [15] theorem 3.1 (with the (3.1.b)″ hypothesis and remark 3.10.3) to obtain a lifting $\hat{\mu} : \hat{X} \times \hat{X} \to \hat{X}$, $\hat{\mu} \mid \hat{X} \vee \hat{X} = \mathscr{F}$ and $(\hat{\phi}_\lambda)^m \circ \hat{\mu} \sim \hat{\mu} \circ (\phi_\lambda \times \phi_\lambda)^m$ where $m = \Pi_{r \leq N(x)} \exp \pi_r(\text{fiber}\, \psi'')$, hence m is a power of p and 3.1 follows. (The fact that ψ'' is a $\hat{\mu} - \mu_0$ H-map implies $H^*(\hat{X}, Q; \mu(p)) = H^*(X, Q; \hat{\mu})$.)

3.2. Lemma. *Let p be an odd prime. Given H-spaces $X, \hat{\mu}, Y, \mu, Y_0, \mu_0$ with self H-maps $\phi : X \to X$, $\phi' : Y \to Y$ and $\phi_0' : Y_0 \to Y_0$. Assume $QH^{\text{even}}(X, Z_p) = 0$, $QH^*(\phi, Z_p) = -1$, $\pi_*(\phi') \otimes Z_p = -1$, $\pi_*(\phi_0') \otimes Z_p = -1$. Let $g : Y \xrightarrow{\approx P-p} Y_0$, $f_0 : X \to Y_0$ be H-maps and $\phi' - \phi_0'$, $\phi - \phi_0'$ maps respectively, i.e.,*

$$g \circ \phi' \sim \phi_0' \circ g; \qquad f_0 \circ \phi \sim \phi_0' \circ f_0.$$

Assuming $\pi_n(\text{fiber } g) = 0$ for n-odd and for $n \geqq N$ then if f_0 lifts to a map $X \to Y$ it lifts to an H-map. Moreover any $\phi^{p^r} - (\phi')^{p^r}$ lifting (i.e., $f : X \to Y$ with $f \circ \phi^{p^r} \sim (\phi')^{p^r} \circ f$) is an H-map.

Proof. By Zabrodsky [15] 3.1 if f_0 lifts it lifts to a $\phi^{p^r} - (\phi')^{p^r}$ map, hence suffices to prove that every such map is an H-map. Obviously one may conveniently assume $r = 0$.

Given a $\phi - \phi'$ map $f : X \to Y$ consider its H-deviation $HD(f) = HD(f, \hat{\mu}, \mu) : X \wedge X \to Y$. As g and f_0 are H-maps $f_0 \sim g \circ f$, $g \circ HD(f) \sim *$ and $HD(f) \sim j \circ w$, $w : X \wedge X \to V = \text{fiber } g$, $j : V \to Y$. Decompose V as follows:

$$* = V^{(m)} \to V^{(m-1)} \to \cdots \to V^{(k)} \to V^{(k-1)} \to \cdots \to V^{(0)} = V$$

$V^{(k)}$ $m_k - 1$ connected, $V^{(k+1)} \to V^{(k)}$ is the fiber of $r_k : V^{(k)} \to K(G_k, m_k)$, $G_k = \pi_{m_k}(V^{(k)}) \otimes Z_p$, $\pi_{m_k}(r_k)$ the obvious morphism. Note that by assumption m_k are even.

Suppose inductively that $HD(f)$ lifts to $w_k : X \wedge X \to V^{(k)}$, $HD(f) = j_k \circ w_k$, $j_k : V^{(k)} \to Y$. Now, $HD(f) \circ \Lambda = D_{\mu \circ (f \times f), f \circ \hat{\mu}}$ and as ϕ, ϕ' are H-maps and f is a $\phi - \phi'$ map one has: $HD(f) \circ (\phi \wedge \phi) = \phi' \circ HD(f)$.

ϕ', ϕ_0' induce self maps $\phi_k' : V^{(k)} \to V^{(k)}$ and j_k is a $\phi_k' - \phi'$ map as well as an H-map. Hence:

$$\begin{aligned} * &= D_{HD(f) \circ (\phi \wedge \phi), \phi' \circ HD(f)} = D_{j_k \circ w_k \circ (\phi \wedge \phi), \phi' \circ j_k \circ w_k} \\ &= D_{j_k \circ w_k \circ (\phi \wedge \phi), j_k \circ \phi_k' \circ w_k} = j_k D_{w_k \circ (\phi \wedge \phi), \phi_k' \circ w_k} \end{aligned}$$

and one can replace w_k by

$$\hat{w}_k = w_k + \frac{p-1}{2} D_{w_k \circ (\phi \wedge \phi), \phi_k' \circ w_k}.$$

Now, $\pi_*(\phi') \otimes Z_p = -1 = \pi_*(\phi_0') \otimes Z_p$ implies (possibly after some p^s iterations) $\pi_*(\phi_k) \otimes Z_p = -1$ and hence $-r_k = r_k \circ \phi_k'$.

$$r_k \circ \hat{w}_k = r_k \circ w_k + \frac{p-1}{2}[r_k \circ w_k \circ \phi \wedge \phi - r_k \circ \phi_k' \circ w_k]$$

$$= r_k \circ w_k + \frac{p-1}{2}[r_k \circ w_k \circ \phi \wedge \phi + r_k \circ w_k],$$

$$r_k \circ w_k \in H^{m_k}(X \wedge X, G_k) = H^{m_k}(X \wedge X, Z_p) \otimes G_k.$$

As $QH^{\text{even}}(X, Z_p) = 0$, $QH^*(\phi, Z_p) = -1$ it follows that $H^{\text{even}}(\phi \wedge \phi, Z_p) = 1 = H^{m_k}(\phi \wedge \phi, G_k)$ and $r_k \circ w_k \circ (\phi \wedge \phi) = r_k \circ w_k$.

Hence $r_k \circ \hat{w}_k = pr_k \circ w_k = 0$ and $\hat{w}_k$ lifts to $w_{k+1} : X \wedge X \to V_{k+1}$. $V_m = *$ implies 3.2.

3.3. Proof of Theorem 2. Given X, μ a new multiplication $\hat{\mu}$ as in 3.1 for $p = 3$ and $\lambda = -1$: ϕ_{-1}—the multiplicative inverse of μ (and essentially of $\hat{\mu}$ as well). As $H^*(X, Q, \mu) = H^*(X, Q, \hat{\mu})$ and $H^*(X, Z)$ is 3-torsion free $H^*(X, Z_3; \mu) = H^*(X, Z_3, \hat{\mu})$ as a Hopf algebra, $QH^*(\phi_{-1}, Z_3) = -1 = PH^*(\phi_{-1}, Z_3)$.

Let $x \in PH^{6n+2a-1}(X, Z_3)$ be realized by an H-map $f_0 : X \to K(Z_3, 6n + 2a - 1) = K_0$. Obviously $f_0 \circ \phi_{-1} \sim \phi_0' \circ f_0$, $\phi_0' = -1 : K_0 \to K_0$. Let E_1 be the space in $(A)_N$ (the proof of Theorem 1) for $N = 6n + 2a - 1$, and replace n by $3n$. Let $\phi' : E_1 \to E_1$ be the multiplicative inverse of the loop multiplication in E_1. As fiber $E_1 \to K_0$ is $K(18n + 2a - 2, Z_3)$, by 3.2, if f_0 lifts to $f_1 : X \to E_1$ it lifts to an H-map. f_0 lifts if and only if $\mathcal{P}^{3n}x = 0$. $E_2 \to E_1$ in $(A)_N$ is the fibration included by an H-map $E_1 \to K(Z_3, 18n + 6a - 2)$ and as $PH^{18n+6a-2}(X, Z_3) = 0$ f_1 lifts to $f_2 : X \to E_2$.

$$HD(f_2) = j_2 \cdot w, \qquad w : X \wedge X \to K(Z_3, 18n + 6a - 3)$$

and

$$\bar{\hat{\mu}}^* f_2^* y = (f_2^* \otimes f_2^*)\bar{\mu}^* v + \mathcal{P}^{3-a} w,$$

hence:

(α) $a = 2$, $|x| = 6n + 3$

$$(\alpha.1) \qquad \bar{\hat{\mu}}^* f_2^* y = \mathcal{P}^1 x \cdot x \otimes x + x \otimes \mathcal{P}^1 x \cdot x + \mathcal{P}^1 w$$

reduce $H^*(X, Z_3)$ by the ideal I generated by $\bigoplus_{m > 6n+7} PH^m(X, Z_3)$.

(β) $a = 1$, $|x| = 6n + 1$

$$(\beta.1) \qquad \begin{aligned} \bar{\hat{\mu}}^* f_2 y &= \mathcal{P}^2 x \cdot x \otimes x + x \otimes \mathcal{P}^2 x \cdot x \\ &+ x \cdot \mathcal{P}^1 x \otimes \mathcal{P}^1 x + \mathcal{P}^1 x \otimes x \cdot \mathcal{P}^1 x + \mathcal{P}^2 w \end{aligned}$$

and reduce by the ideal I generated by $\bigoplus_{m>6n+5} PH^m(X, Z_3)$ (and (β.1) thus loses its first two terms in the left hand side of the equation).

In both cases if $\mathcal{P}^1 x \not\subset \operatorname{im} \mathcal{P}^2$ the quotient algebra $H^*(X, Z_3)/I$ has the form $A \otimes \Lambda(x, \mathcal{P}^1 x)$ as an algebra over $Z_p[\mathcal{P}^1]/(\mathcal{P}^1)^3 \subset \mathcal{A}(3)$. Reducing $\operatorname{mod} A$ one will have (α.1) or (β.1) holding in $\Lambda(x, \mathcal{P}^1 x)$ which is impossible.

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